Lecture 8

Last time: $(\mathfrak{g}, \mathfrak{h}) \rightsquigarrow (V, \langle , \rangle, \Phi)$

roots, eigenvals of $\mathrm{ad}_{\mathfrak{h}}$.

Killing form

semisimp — Cartan — $\mathbb{R}$ form of $\mathfrak{h}^*$

Key properties: $\alpha, \beta \in \Phi \Rightarrow \quad p(\alpha,\beta) = \dfrac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$

and $\quad S_\alpha(\beta) = \beta - \dfrac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \in \Phi$

Φ spans but is not lin indep.

Introduce a notion of positivity on V using a bases $v_1 \cdots v_k$.
Say "$\varphi > 0$" if the first nonzero coef in φ (rel v_i) is positive.

Thus $\varphi \neq 0 \Rightarrow$ either $\varphi > 0$ or $\varphi < 0$. (No "ties")

$\Phi = \Phi^+ \cup -\Phi^+$ where $\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}$

Now let $\Delta = \{\alpha \in \Phi^+ \mid \nexists\ \beta_1, \beta_2 \in \Phi^+ \text{ with } \alpha = \beta_1 + \beta_2\}$
the "indecomposable positive" roots. Simple roots is the standard term.

Thm. Δ is a basis of V. $\quad (|\Delta| = \dim V = k = \text{rank of } \mathfrak{g})$
$\Phi \subset \mathbb{Z}_\Delta$, $\Phi^+ \subset \mathbb{Z}^+_\Delta$.

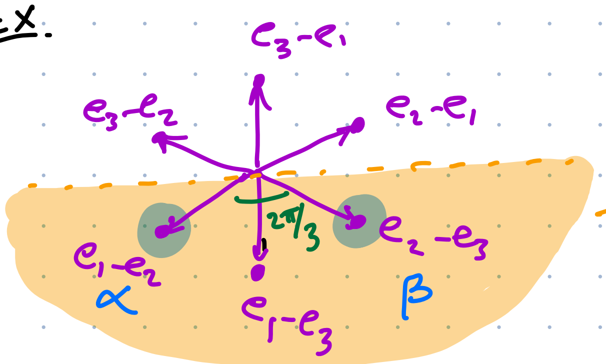
Now we form a graph: vertices are Δ.

edges from α to β is $p(\alpha,\beta)\,p(\beta,\alpha)$

If #edges > 1 ($\Leftrightarrow \langle\alpha,\alpha\rangle \neq \langle\beta,\beta\rangle$) then the group of edges is oriented toward the shorter root.

Thm $\mathfrak{g}$ simple iff this graph is connected.
Else, the components correspond to its simple summands.

Ex.



$\Delta = \{\alpha, \beta\}$ $\langle \alpha, \alpha\rangle = \langle \beta,\beta\rangle = 1$, say

$\langle \alpha, \beta\rangle = \cos \frac{2\pi}{3} = -\frac{1}{2}$

$p(\alpha,\beta) = \frac{2(-\frac{1}{2})}{1} = -1 \quad p(\beta,\alpha) = -1$

$p(\alpha,\alpha) = p(\beta,\beta) = 2.$

Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \rightsquigarrow$ •—• Dynkin diagram. — name for the graph thus obtained.

Ex. $Sp(2n,\mathbb{C}) = \{A \in GL_{2n}\mathbb{C} \mid A^T J A = A\}$ where $J = \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$ also called $Sp(n,\mathbb{C})$

$\mathfrak{sp}(2n,\mathbb{C}) = \{X \in \mathfrak{gl}_{2n}\mathbb{C} \mid X^T J + JX = 0\}$ is simple.

$= \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \text{ where } B^T = B,\ C^T = C \right\}.$

Let $n=2$. $Sp(4,\mathbb{C})$. $\mathfrak{h} = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & -t_1 & \\ & & & -t_2 \end{pmatrix} \middle| t_1, t_2 \in \mathbb{C} \right\}$. rank $= 2$.

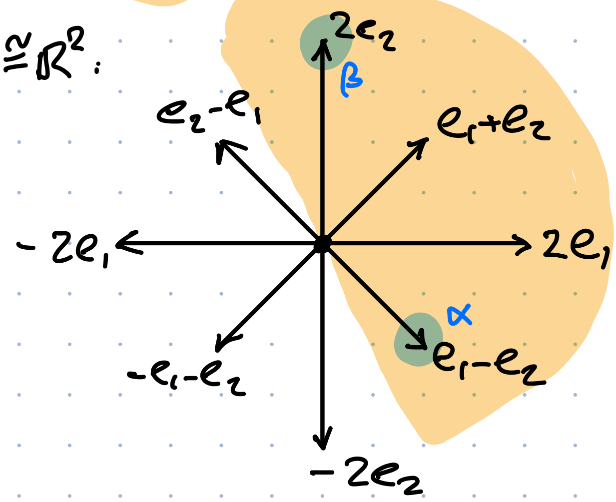
$e_i(t) = t_i$ e_1, e_2 basis $\mathfrak{h}^*$

$\dim \mathfrak{g} = 10$ Basis for compl of $\mathfrak{h}$:

$\{ a_{12} = 1,\ a_{21} = 1,\ b_{11} = 1,\ b_{22} = 1,\ c_{11} = 1,\ c_{22} = 1,\ c_{12} = c_{21} = 1,\ b_{12} = b_{21} = 1 \}$

Roots: $e_1 - e_2 \quad e_2 - e_1 \quad 2e_1 \quad 2e_2 \quad -2e_1 \quad -2e_2 \quad -e_1 - e_2 \quad e_1 + e_2$

$V \cong \mathbb{R}^2$:



$\Phi^+ = \{e_1 - e_2,\ 2e_1,\ 2e_2,\ e_1 + e_2\}$

$\Delta = \{e_1 - e_2, 2e_2\} = \{\alpha, \beta\}$

$\Phi^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta\}$

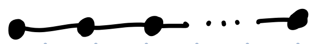
$\langle \alpha, \alpha \rangle = 2 \quad \langle \beta, \beta\rangle = 4. \quad \langle \alpha, \beta \rangle = -2$

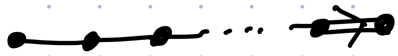
$p(\alpha,\beta) = \frac{2(-2)}{4} = -2 \qquad p(\beta,\alpha) = \frac{2(-2)}{2} = -1$

Cartan $= \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ Dynkin $\underset{\alpha}{\bullet} \Leftarrow \underset{\beta}{\bullet}$ two edges as $p(\alpha,\beta)p(\beta,\alpha) = 2$.

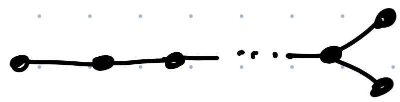
Back to the general classification

The connected graphs obtained this way are: $(n = \text{rank})$

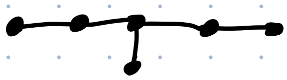
A_n  $\mathfrak{sl}_{n+1}, \mathbb{C}$ $\quad W = \mathrm{Sym}_{n+1}$

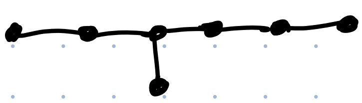
B_n  $\mathfrak{so}(2n+1, \mathbb{C})$ $\quad W = \mathrm{Sym}_n \ltimes (\mathbb{Z}/2)^n$

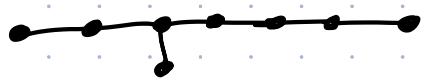
C_n  $\mathfrak{sp}(2n, \mathbb{C})$ $\quad W = \mathrm{Sym}_n \ltimes (\mathbb{Z}/2)^n$

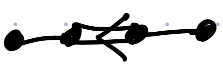
D_n  $\mathfrak{so}(2n, \mathbb{C})$ $\quad W = \mathrm{Sym}_n \ltimes \underbrace{(\mathbb{Z}/2)^{n-1}}$

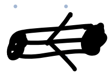
vectors sum to zero in $(\mathbb{Z}/2)^n$.

E_6  $\dim \mathfrak{g} = 78$

E_7  $\dim \mathfrak{g} = 133$

E_8  $\dim \mathfrak{g} = 248$

F_4  $(1:2)$ $\quad \dim \mathfrak{g} = 52$

G_2  $(1:3)$ $\quad \dim \mathfrak{g} = 14 \quad \mathfrak{g} \subset \mathfrak{so}(8, \mathbb{C})$

Note. $SO(k, \mathbb{C}) = \{A \in \mathfrak{gl}_k \mathbb{C} \mid A^T A = I\}$ $\quad \mathfrak{so}(k, \mathbb{C}) = \{X \mid X + X^T = 0\}$

Weyl group $W(V, \langle\rangle, \Phi) = W(\mathfrak{g}, \mathfrak{h})$ = the group generated by the reflections

$$s_\alpha(\varphi) = \varphi - 2\frac{\langle \varphi, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

acting on V or $\mathfrak{h}^*$ (or $\mathfrak{h}$, using $\mathfrak{h} \cong \mathfrak{h}^*$ via B)

Facts. Finite (injects into Sym_Φ)

Acts transitively on possible Φ^+ choices.

Parabolic subalgebras Suppose $\mathfrak{g}, \mathfrak{h}, \Phi^+$ fixed.

$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \subset \mathfrak{g}$ subalgebra. **Solvable** as the coef of Δ get bigger each time you bracket.

Thm. $\mathfrak{b}$ is maximal among solvable subalg.
Any max solv. subalg is conjugate to $\mathfrak{b}$.
means inner aut related.

Let $S \subset \Delta$. Let $\Phi_S = \Phi \cap \mathbb{Z}_S^+$.

$\mathfrak{p}_S := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi_S} \mathfrak{g}_{-\alpha}$.